On Delay-Independent Diagonal Stability of Max-Min Congestion Control

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***Abstract*— Network feedback in a congestion-control system is subject to delay, which can significantly affect stability and performance of the entire system. While most existing stability conditions explicitly depend on delay D_i of individual flow i, a recent study [24] shows that the combination of a symmetric Jacobian A and condition $\rho(A) < 1$ guarantees local stability of the system regardless of D_i. However, the requirement of symmetry is very conservative and no further results have been obtained beyond this point. In this paper, we proceed in this direction and gain a better understanding of conditions under which congestion-control systems can achieve delay-independent stability. Towards this end, we first prove that if Jacobian matrix A satisfies $||A|| < 1$ for any *monotonic* induced matrix norm $||.||$, the system is locally stable under arbitrary diagonal delay D_i. We then derive a more generic result and prove that delay-independent stability is guaranteed as long as A is Schur diagonally stable [9], which is also observed to be a necessary condition in simulations. Utilizing these results, we identify several classes of well-known matrices that are stable under diagonal delays if $\rho(A) < 1$ and prove stability of MKC [24] with arbitrary parameters α_i and β_i.**

I. Introduction

Several max-min congestion control algorithms (e.g., XCP [11], RCP [5], VCP [23], MKC [24], and JetMax [25]) have been recently proposed. These protocols receive feedback from the most-congested router in their path and exhibit appealing performance from both theoretical and practical perspectives. Thus, stability of these systems, especially when delay is present in the network feedback, has recently received a fair amount of attention [4], [7], [13], [14], [16], [19], [20], [21], [22], [24]. However, most existing stability conditions (e.g., [7], [11], [14], [19]) require that parameters of the control equation be adaptively tuned according to feedback delay D_i of user i, making them undesirable in practice due to the resulting unfairness between flows with different RTTs and oscillations when delays are not properly estimated by end-users. Although this limitation is partially resolved by the recent work [24], which proves that max-min systems with a symmetric Jacobian matrix are locally stable regardless of delay, the requirement of symmetry is very restrictive in practice and understanding whether the same result holds for a wider class of matrices remains open.

In this paper, we gain a deeper insight into stability of max-min congestion control systems under diagonal delays. Most max-min systems (e.g., MKC [24], RCP [5], and XCP [11]) can be linearized to the following shape (more on this below):

$$x_i(n) = \sum_{j=1}^{N} a_{ij} x_j(n - D_i), \tag{1}$$

where a_{ij} are some constants, N is the number of flows, and $x_i(n)$ and D_i are, respectively, the sending rate and round-trip time of user i. Using this model, we first present an alternative proof of Theorem 1 in [24] under time-invariant delay D_i, based on which we derive a sufficient stability condition of (1) to be $||A||_2 < 1$, where $A = (a_{ij})$ is the coefficient matrix of the system and matrix norm $||.||_2$ is induced by the L^2 vector norm. Clearly, this condition is more generic than the one obtained in [24], which required A to be real and symmetric.

Subsequently, we prove that this result actually extends to any matrix norm induced by a *monotonic* vector norm (which subsumes all standard vector norms, such as $||.||_1$, $||.||_2$, $||.||_\infty$, and $||.||_\infty^w$). Moreover, we prove that a special norm $||A||_s = \inf_{W \in \mathcal{P}^*} ||WAW^{-1}||_2$ (where $\mathcal{P}^*$ is the set of all positive diagonal matrices) is a monotonic induced norm and further generalize the sufficient stability condition of system (1) to $||A||_s < 1$, whose necessity is also indicated by simulations. Armed with these results, we identify several classes of systems that are stable under diagonal delays if and only if they are stable under undelayed feedback. This finding allows us to prove stability of Max-min Kelly Control (MKC) [24] with arbitrary parameters α_i and β_i. We also discuss and verify obtained results using Matlab simulations.

The rest of the paper is organized as follows. In Section II, we describe modeling assumptions of this paper and review existing work in the area of delay-independent stability. In Section III, we present our main results of the paper and verify them via Matlab simulations. In Section IV, we conclude the paper and suggest directions for future work.

II. Background

A. Modeling Single-Link Congestion Control

Assume a generic model of max-min congestion control that is composed of M links and N users. Each flow i solicits feedback p_i from a particular link, which we call the *bottleneck* link of user i, in its path. Assuming that bottleneck assignments of the network are fixed [16], [22], it is proven in [25] that the entire system can be decomposed into subsystems of users constrained by individual bottlenecks, which is stable independent of delays if and only if individual subsystems are. Thus, in what follows we restrict our focus to congestion control in single-link networks and note that

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the obtained results are applicable to all multi-link max-min systems with time-invariant bottleneck assignments.

Consider a network with N users accessing a single bottleneck link. Feedback delays arise from transmission, propagation, and queuing delay at individual links. Specifically, the time lag for a packet to travel from sender i to its bottleneck link is denoted by forward delay $D_i^{\rightarrow}$, while delay from the router to the receiver and subsequently from the receiver back to the sender is denoted by backward delay $D_i^{\leftarrow}$. Clearly, summation of the forward and backward delays forms the round-trip delay D_i of user i, i.e., $D_i = D_i^{\rightarrow} + D_i^{\leftarrow}$.

Each acknowledgement packet of flow i carries certain network feedback $p(n)$, which is continually computed by the bottleneck router as a function of the combined incoming rate of all flows, i.e.,

$$p(n) = g\Big(\sum_{j=1}^{N} x_j(n - D_i^{\rightarrow})\Big). \tag{2}$$

This feedback is utilized by each source i to update its sending rate $x_i(n)$ according to the following control rule:

$$x_i(n) = f_i\big(p(n - D_i^{\leftarrow})\big), \tag{3}$$

where function $f_i(.)$ is assumed to be differentiable in the equilibrium point.

Note that (2)-(3) usually forms a nonlinear system, whose local stability can be studied by linearizing the system in the equilibrium point $\mathbf{x}^*$. Denote by $A = (a_{ij})$ the corresponding Jacobian matrix, then the linearized system is described as follows:

$$x_i(n) = \sum_{j=1}^{N} a_{ij} x(n - D_j^{\rightarrow} - D_i^{\leftarrow}), \tag{4}$$

where $a_{ij} = \partial f_i / \partial x_j |_{\mathbf{x}^*}$.

The following result transforms stability analysis of (4) to that of an equivalent system (1).

Lemma 1: System (4) is stable under all heterogeneous directional delays $D_i^{\rightarrow}$ and $D_i^{\leftarrow}$ if and only if system (1) is stable under all round-trip delays D_i.

We omit the proof of this lemma for brevity and note that a similar result is derived for continuous-time systems in [14]. Compared to (4), system (1) has a simpler shape and is more amenable to analysis. Thus, in the rest of paper, we focus our study on system (1) and keep in mind that our results also apply to (4).

B. Stability Results

In this paper, we are interested in delay-independent stability as defined below of a given dynamical system.

Definition 1: We call a system *stable independent of delays* if neither its control gain nor stability condition explicitly involves delays.

It is well-known that system (1) under zero delay is stable if and only if the spectral radius $\rho(A) < 1$ [6]. When delay is introduced into the control loop, stability analysis of the resulting system becomes more complicated. The most recent result in this direction is presented in [24], which proves that (1) with a *symmetric* Jacobian A is stable regardless of delay if and only if $\rho(A) < 1$.

One more generic version of this problem is to study stability of the system under arbitrary delay $D_{ij} > 0$, i.e.,

$$x_i(n) = \sum_{j=1}^{N} a_{ij} x_j(n - D_{ij}). \tag{5}$$

We note that in the last equation each feedback is delayed by D_{ij} time units instead of a round-trip delay D_i as in (1). Thus, stability conditions of system (5) are sufficient, but not necessary for system (1).

The convergence property of (5) is initially studied by Chazan and Miranker [3] in the context of asynchronous iteration and the first sufficient and necessary condition is due to Bertsekas and Tsitsiklis [2], who prove that (5) is stable for all uniformly-bounded and time-varying delays $D_{ij}(n)$ if $\rho(|A|) < 1$ and unstable under a certain set of delays $D_{ij}(n)$ if $\rho(|A|) \geq 1$. The same result is later obtained by Kaszkurewicz *et al.* [8] using a different technique.

A slightly stronger version of this result is available in [18], which first introduces the following terminology.

Definition 2 ([18]): Time delays $D_{ij}(n)$ are *admissible* if:

$$\lim_{n \to \infty} n - D_{ij}(n) = \infty, \ \forall i, j \tag{6}$$

and *regulated* (uniformly bounded) if $0 \leq D_{ij}(n) < \bar{D}, \ \forall i, j$ for some non-negative constant $\bar{D}$ that is independent of n.

Then, Su *et al.* [18] prove that system (5) is stable for all admissible delays $D_{ij}(n)$ when $\rho(|A|) < 1$ and unstable for a certain sequence of regulated delays $D_{ij}(n)$ with $\bar{D} = 1$ when $\rho(|A|) \geq 1$.

In addition, Kaszkurewicz *et al.* provide an alternative way of verifying condition $\rho(|A|) < 1$ by showing that it is equivalent to the existence of a positive diagonal matrix W such that $||W^{-1}AW||_\infty < 1$. This is a consequence of [10, Appendix 2]:

$$\rho(|A|) = \inf_{W \in \mathcal{P}^*} ||W^{-1}AW||_\infty, \tag{7}$$

where $\mathcal{P}^*$ is the set of all positive, diagonal matrices.

However, for system (1), condition $\rho(|A|) < 1$ is too strong and is not necessary. One example is given in [24], which demonstrates that MKC in the form of (1) may be stable even though $\rho(|A|) > 1$. The relationship between stability conditions of (1) and (5) under different types of delays is illustrated in Fig. 1. As shown in the figure, stability of the system under zero delay and arbitrary delays D_{ij} has been well studied; however, understanding of (1) under diagonal delays D_i is lacking in the current picture. Thus, in the rest of this paper, we fill in this void and investigate conditions under which system (1) achieves delay-independent stability.

III. Main Results

A. Induced Matrix Norms

We start by recalling definitions and properties of vector-induced matrix norms, which are used later in the paper.

Definition 3 ([6]): Matrix norm $||A||$ is a non-negative number with the following properties:

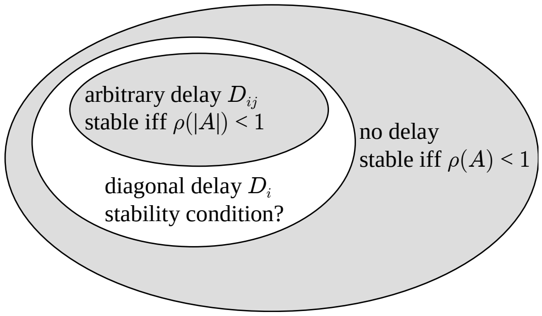


Fig. 1. Illustration of the current research status of delay-independent stability of system (1) under different types of delays.

1) $||A|| > 0$ when $A \neq 0$, $||A|| = 0$ if and only if $A = 0$.
2) $||\beta A|| = |\beta| \cdot ||A||$.
3) $||A + B|| \leq ||A|| + ||B||$.
4) $||AB|| \leq ||A|| \cdot ||B||$.

One special class of matrix norms, called *induced matrix norms*, is defined as follows.

Definition 4 ([6]): Matrix norm $||.||$ induced by or subordinate to a given vector norm $||.||$ is defined as following:

$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}. \tag{8}$$

The following properties of induced matrix norms are available from [6].

Property 1: Any induced matrix norm $||.||$ satisfies inequality $||Ax|| \leq ||A|| \cdot ||x||$.

Property 2: For any matrix A and an arbitrary induced matrix norm $||.||$, we have $\rho(A) \leq ||A||$.

To better understand Definition 4, consider the following commonly used induced matrix norms: spectral norm $||A||_2 = \sqrt{\rho(A^*A)}$ (where A^* is the conjugate transpose of A) induced by the L^2 vector norm, maximum absolute column sum norm $||A||_1 = \max_j(\sum_i |a_{ij}|)$ induced by the L^1 vector norm, maximum absolute row sum norm $||A||_\infty = \max_i(\sum_j |a_{ij}|)$ induced by the L^∞ vector norm, and weighted maximum norm $||A||_\infty^w = \max_i(\sum_j |a_{ij}|w_j)/w_i$ ($w > 0$) induced by the weighted infinity vector norm $||x||_\infty^w$. We refer interested readers to [6] for more details.

B. Alternative Proof of Theorem 1 in [24]

Leveraging definitions and properties of induced matrix norms, we next present an alternative proof of [24, Theorem 1] with much simpler manipulations. This proof further leads us to more generic results derived in Section III-C.

Theorem 1: If A is symmetric and stable, system (1) is stable regardless of delays D_i.

Proof: Applying the z-transform to system (1), we obtain:

$$\mathbf{H}(z) = ZA\mathbf{H}(z), \tag{9}$$

where $Z = \mathrm{diag}(z^{-D_i})$ is a diagonal matrix and $\mathbf{H}(z)$ is the vector of z-transforms of each flow rate x_i:

$$\mathbf{H}(z) = \big(H_1(z), H_2(z), \cdots, H_N(z)\big)^T. \tag{10}$$

Notice that system (1) is stable if and only if all poles of its z-transform $\mathbf{H}(z)$ are within the unit circle in the z-plane. To examine this condition, re-organize the terms in (9):

$$(ZA - I)\,\mathbf{H}(z) = 0. \tag{11}$$

Next notice that the poles of $\mathbf{H}(z)$ are simply the roots of:

$$\det(ZA - I) = 0. \tag{12}$$

Thus, ensuring that all roots of (12) are inside the open unit circle will be both sufficient and necessary for system (1) to be stable. Bringing in notation $\mathcal{F}(z) = \det(ZA - I)$, we can re-write $\mathcal{F}(z)$ as following:

$$\begin{aligned} \mathcal{F}(z) &= \det(Z[A - Z^{-1}I]) \\ &= \det(Z)\det(A - Z^{-1}). \end{aligned} \tag{13}$$

Noticing that $\det(Z)$ is strictly non-zero for non-trivial z, we can reduce (12) to:

$$\mathcal{F}(z) = \det\big(A - Q(z)\big) = 0, \tag{14}$$

where $Q(z) = \mathrm{diag}(z^{D_i})$.

To prove that all roots of (14) lie in the open unit circle, we suppose in contradiction that there exists a root $|z_0| \geq 1$ such that $\mathcal{F}(z_0) = 0$. Denote by B matrix $Q(z_0)$. Following [15] and using basic matrix algebra, it is easy to have that there exists a non-zero vector v such that $Av = Bv$. For symmetric matrices, we can write $||A||_2 = \rho(A) < 1$ and:

$$||A||_2 = \sup_{x \neq 0} \frac{||Ax||_2}{||x||_2} \geq \frac{||Av||_2}{||v||_2} = \frac{||Bv||_2}{||v||_2}, \tag{15}$$

where $||.||_2$ in application to vectors is a standard L^2 norm.

Since B is diagonal with $|b_{ii}| = |z_0|^{D_i} \geq 1$, Bv is simply a vector $(v_1 b_{11}, \ldots, v_N b_{NN})^T$ and we can express vector norm L^2 using the following:

$$\frac{||Bv||_2}{||v||_2} = \frac{\Big(\sum_{i=1}^N |v_i|^2 |b_{ii}|^2\Big)^{1/2}}{\Big(\sum_{i=1}^N |v_i|^2\Big)^{1/2}} \geq \frac{\Big(\sum_{i=1}^N |v_i|^2\Big)^{1/2}}{\Big(\sum_{i=1}^N |v_i|^2\Big)^{1/2}} = 1. \tag{16}$$

Thus, we get that both $||A||_2 \geq 1$ and $||A||_2 < 1$ must be satisfied simultaneously, which is a contradiction. This means that no $|z_0| \geq 1$ can be a root of $\mathcal{F}(z)$ and that any heterogenous systems with a symmetric stable matrix A is stable under arbitrary delay. ■

Note that the above reasoning indicates symmetry of A is not necessary and leads us to the following generalization.

Corollary 1: If Jacobian A satisfies $||A||_2 < 1$, system (1) is stable for all delays D_i.

C. Weaker Sufficient Conditions

We next extend the requirement of L^2 norm in the last result to any *monotonic* vector norm, which is defined below.

Definition 5 ([1]): If a vector norm $||.||$ on R^n satisfies the following inequality:

$$||x|| \leq ||y|| \text{ for all } x, y \in R^n \text{ such that } |x| \leq |y|, \tag{17}$$

we call this norm *monotonic*.

This allows us to prove the following theorem.

Theorem 2: If there exists a *monotonic* vector norm $||.||_\alpha$ such that induced matrix norm $||A||_\alpha < 1$, system (1) is stable regardless of delays D_i.

Proof: We utilize the technique in Theorem 1, whose goal is to show that $||Bv||_\alpha/||v||_\alpha \geq 1$ for all v. Again, notice that $Bv = (v_1 b_{11}, \ldots, v_N b_{NN})^T$ and that $|b_{ii}| \geq 1$. Due to the monotonicity of the norm and the fact that $|v_i| \leq |b_{ii} v_i|$, we directly get $||v||_\alpha \leq ||Bv||_\alpha$. The remaining proof is similar to that of Theorem 1. ∎

The following result provides a systematic way for generating monotonic induced matrix norms.

Theorem 3: Matrix norm $||A||_2^w = ||WAW^{-1}||_2$ for any non-singular diagonal matrix $W = \text{diag}(w)$ is a monotonic induced norm.

Proof: First write:

$$||A||_2^w = \sup_{x\neq 0} \frac{||Ax||_2^w}{||x||_2^w} = \sup_{x\neq 0} \frac{||WAx||_2}{||Wx||_2}. \quad (18)$$

Next, we need to show that $||x||_2^w = ||Wx||_2$ is monotonic with respect to x. In other words, we must show that for any two vectors x_1 and x_2 such that $|x_1| \leq |x_2|$, $||Wx_1||_2 \leq ||Wx_2||_2$. This directly follows from the fact that $|Wx_1| \leq |Wx_2|$ for any non-singular diagonal W and from monotonicity properties of $||.||_2$ in application to vectors. ∎

Then, a more generic sufficient stability condition is easy to derive from Theorem 2.

Corollary 2: The following is sufficient for system (1) to be stable for all delays D_i:

$$||A||_s = \inf_{W\in\mathcal{P}^*} ||WAW^{-1}||_2 < 1, \quad (19)$$

where $\mathcal{P}^*$ is the set of all positive diagonal matrices.

We next show that spectral norm in (19) is weaker than infinity norm, which is used in (7) as the sufficient and necessary condition for stability of system (5).

Theorem 4: For any matrix A, we have $||A||_s \leq \rho(|A|) = \inf_{W\in\mathcal{P}^*} ||WAW^{-1}||_\infty$.

Proof: Denote by $D = |A|$ the absolute value of A and observe that:

$$\begin{aligned} ||WAW^{-1}||_2 &= \sqrt{\rho(W^{-1}A^*WWAW^{-1})} \\ &= \sqrt{\rho(A^*W^2AW^{-2})} \\ &\leq \sqrt{\rho(D^*W^2DW^{-2})} \\ &= ||WDW^{-1}||_2, \quad W \in \mathcal{P}^* \end{aligned} \quad (20)$$

where A^* is the conjugate transpose of A. Next, since D is a non-negative matrix, we immediately obtain from [9, Lemma 2.7.25] that:

$$\inf_{W\in\mathcal{P}^*} ||WDW^{-1}||_2 = \rho(D). \quad (21)$$

Completing the chain of arguments, we get:

$$\inf_{W\in\mathcal{P}^*} ||WAW^{-1}||_2 \leq \inf_{W\in\mathcal{P}^*} ||WDW^{-1}||_2 = \rho(|A|) \quad (22)$$

for all matrices A. ∎

D. Delay-Independent Stable Matrices

In this subsection, we apply results obtained so far and identify several classes of matrices that are stable under diagonal delays if and only if they are stable under zero delay, i.e., $\rho(A) < 1$.

We first examine the class of normal matrices $\mathcal{N}$, which are defined as the set of matrices A for which $AA^* = A^*A$, where A^* is the conjugate transpose of A. Normal matrices include symmetric (i.e., $a_{ij} = a_{ji}$), skew-symmetric (i.e., $a_{ij} = -a_{ji}$), Hermitian (i.e., $A^* = A$), skew-Hermitian (i.e., $A^* = -A$), circulant, and unitary matrices (i.e., $A^* = A^{-1}$).

Lemma 2: If $A \in \mathcal{N}$, A is stable under diagonal delays D_i if and only if $\rho(A) < 1$.

Proof: First notice that if matrix A is normal, then A and A^* have the same eigenvectors and their eigenvalues are conjugates of each other [17]. Then applying eigen decomposition on both matrices, we have $A = \Gamma\Lambda\Gamma^{-1}$ and $A^* = \Gamma\Lambda^*\Gamma^{-1}$, where Λ and Λ^* are, respectively, diagonal matrices of eigenvalues of A and A^* and Γ is the matrix with the corresponding eigenvectors. Then, we have:

$$\begin{aligned} ||A||_2 &= \sqrt{\rho(A^*A)} = \sqrt{\rho(\Gamma\Lambda\Gamma^{-1}\Gamma\Lambda^*\Gamma^{-1})} \\ &= \sqrt{\rho(\Gamma\Lambda\Lambda^*\Gamma^{-1})} = \sqrt{\rho^2(A)} = \rho(A). \end{aligned} \quad (23)$$

The rest of proof directly follows from Theorem 1. ∎

We next define $\mathcal{DN}$ as the set of matrices diagonally similar to $\mathcal{N}$. In other words, for any matrix $A \in \mathcal{DN}$, there exists matrix $B \in \mathcal{N}$ and non-singular diagonal matrix W such that $WAW^{-1} = B$. Then, we can prove the result below.

Lemma 3: If $A \in \mathcal{DN}$, A is stable under diagonal delays D_i if and only if $\rho(A) < 1$.

Proof: Let $WAW^{-1} = B$, where B is normal and W is non-singular diagonal. Then, we have $\rho(A) = \rho(WAW^{-1}) = ||WAW^{-1}||_2 = ||A||_2$. According to Corollary 1, this implies that $\rho(A) < 1$ is both sufficient and necessary for A to be stable under delays D_i. ∎

The third class is $\mathcal{P}$, which consists of non-negative/non-positive matrices (i.e., $A \geq 0$ or $A \leq 0$). Combining the facts that $\rho(A) = \rho(|A|)$ and $\rho(A) \leq ||A||_s \leq \rho(|A|) = \rho(A)$ and invoking Corollary 2, we directly arrive at the following lemma.

Lemma 4: If $A \in \mathcal{P}$, A is stable under diagonal delays D_i if and only if $\rho(A) < 1$.

Similar to $\mathcal{DN}$, we define $\mathcal{DP}$ as the set of matrices diagonally similar to $\mathcal{P}$. Then, we have the following result.

Lemma 5: If $A \in \mathcal{DP}$, A is stable under diagonal delays D_i if and only if $\rho(A) < 1$.

Proof: Assume A is diagonally similar to a non-negative/non-positive matrix B. Then, $B = WAW^{-1}$ for some non-singular diagonal matrix W. Noticing that $\rho(|B|) = \rho(B) = \rho(A)$ and $\rho(A) \leq ||A||_s \leq \rho(|A|) = \rho(B) = \rho(A)$, we have $\rho(A) = ||A||_s$, which directly follows from Corollary 2 that condition $\rho(A) < 1$ is both sufficient and necessary for system (1) to be stable. ∎

Next, define radial matrices $\mathcal{R}$ as the class of matrices satisfying $||A||_2 = \rho(A)$. Recalling that $||.||_2$ is induced from

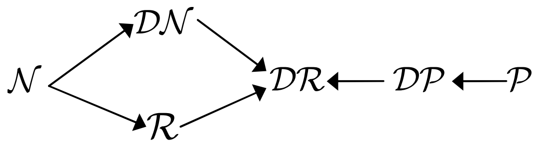


Fig. 2. Relationship between various classes of matrices

the L^2 vector norm and invoking Theorem 2, the following lemma is obvious.

Lemma 6: If $A \in \mathcal{R}$, A is stable under diagonal delays D_i if and only if $\rho(A) < 1$.

Analogous to Lemma 3, the last result also applies to $\mathcal{DR}$, which denotes any matrix diagonally similar to $\mathcal{R}$. Results obtained in this subsection are summarized as following.

Theorem 5: The following matrices are stable under arbitrary diagonal delays D_i if and only if $\rho(A) < 1$: $\mathcal{N}$, $\mathcal{DN}$, $\mathcal{P}$, $\mathcal{DP}$, $\mathcal{R}$, and $\mathcal{DR}$.

We conclude by identifying the relationship between different classes of matrices examined in this subsection.

Theorem 6: The following relations hold: $\mathcal{N} \subset \mathcal{DN} \subset \mathcal{DR}$, $\mathcal{N} \subset \mathcal{R} \subset \mathcal{DR}$, and $\mathcal{P} \subset \mathcal{DP} \subset \mathcal{DR}$.

Proof: We only present proofs of $\mathcal{DN} \subset \mathcal{DR}$ and $\mathcal{DP} \subset \mathcal{DR}$ and omit others for brevity. First let $A \in \mathcal{DN}$, then there exists non-singular diagonal matrix W such that $WAW^{-1} = B \in \mathcal{N}$. Since $\mathcal{N} \subset \mathcal{R}$ according to (23), we have $B \in \mathcal{R}$ and therefore $\mathcal{DN} \subset \mathcal{DR}$.

We next prove $\mathcal{DP} \subset \mathcal{DR}$. Since $A \in \mathcal{DP}$, there exists diagonal matrix W and $B \in \mathcal{P}$ such that $B = WAW^{-1}$. From (21), we know that for any $B \in \mathcal{P}$, there exists diagonal matrix V such that $||VBV^{-1}||_2 = \rho(B)$. Letting $C = UAU^{-1}$ and $U = WV$, we have $||C||_2 = \rho(B) = \rho(A) = \rho(C)$, which implies that A is diagonally similar to radial matrix C and therefore $\mathcal{DP} \subset \mathcal{DR}$. ■

This result is also illustrated in Fig. 2, where notation $A \to B$ refers to $A \subset B$. As seen in the figure, $\mathcal{DR}$ is the widest class of matrices for which $\rho(A) < 1$ guarantees diagonal stability of (1). We leave exploration of more generic classes of delay-independent matrices for future work.

E. Application to MKC

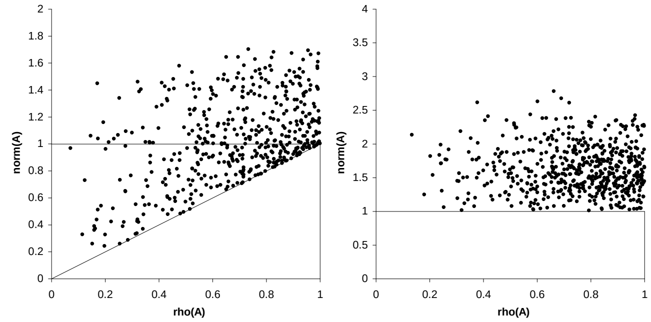
Recall that delay-independent stability of Max-min Kelly Control (MKC) has been established in [24], in which the proof holds only for a symmetric Jacobian matrix under the assumption of constant parameters α and β. We next utilize the techniques developed in this paper to prove MKC's stability under arbitrary parameters α_i and β_i. The resulting control equation thus becomes:

$$x_i(n) = x_i(n - D_i) + \alpha_i - \beta_i p(n) x_i(n - D_i), \qquad (24)$$

where feedback $p(n)$ is a function of the combined incoming rate of all flows.

Theorem 7: Assuming $p(n)$ is differentiable, system (24) is stable under arbitrary delays D_i if:

$$0 < \beta_i \Big(p^* + \sum_{k=1}^{N} x_i^* p'\Big) < 2, \quad i = 1, \dots, N, \qquad (25)$$



(a) stable (b) unstable

Fig. 3. Delayed stability as a function of $\rho(A)$ and $||A||_2$.

where x_i^* and p^* are, respectively, the equilibrium points of $x_i(n)$ and $p(n)$, and p' is the derivative of $p(n)$ evaluated in the equilibrium point.

Proof: Linearizing system (24) in the equilibrium point $\mathbf{x}^*$, we get the Jacobian matrix $A = (a_{ik})$ as follows:

$$a_{ik} = \begin{cases} -\beta_i x_i^* p' & i \neq k \\ 1 - \beta_i (p^* + x_i^* p') & i = k \end{cases}. \qquad (26)$$

Introducing diagonal matrix $W = \text{diag}(\sqrt{\beta_k x_k^*})$, we can construct a new matrix $B = (b_{ik}) = WAW^{-1}$ given below:

$$b_{ik} = \begin{cases} -\sqrt{\beta_i x_i^* \beta_k x_k^*} p' & i \neq k \\ 1 - \beta_i (p^* + x_i^* p') & i = k \end{cases}, \qquad (27)$$

which is symmetric. Thus, matrix A is diagonally similar to a symmetric matrix B and, according to Lemma 3, is stable under diagonal delays D_i if and only if $\rho(A) < 1$.

We next define square matrix $C = (c_{ik})$ such that:

$$c_{ik} = \begin{cases} \beta_i x_i^* p' & i \neq k \\ \beta_i (p^* + x_i^* p') & i = k \end{cases}. \qquad (28)$$

It is easy to see that $A = I - C$, where I is the identity matrix. Applying eigen decomposition on matrix C, we re-write C as $C = \Gamma \Lambda \Gamma^{-1}$, where Γ is a matrix of eigenvectors of C and Λ is a diagonal matrix with the corresponding eigenvalues. Then, we can compute the spectral radius $\rho(A)$ as follows:

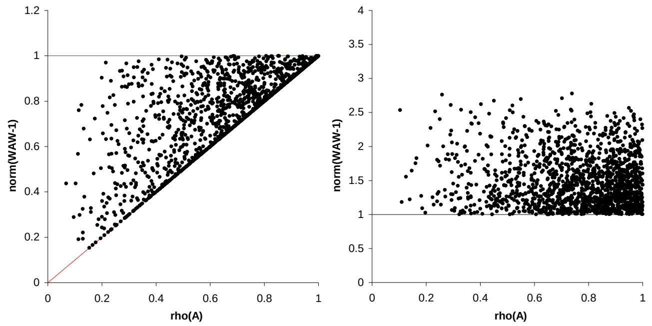
$$\begin{aligned} \rho(A) &= \rho(I - \Gamma \Lambda \Gamma^{-1}) = \rho(\Gamma (I - \Lambda) \Gamma^{-1}) \\ &= \rho(I - \Lambda) = 1 - \rho(C). \end{aligned} \qquad (29)$$

Then, condition $\rho(A) < 1$ leads to $0 < \rho(C) < 2$. Combining Property 2 and the assumption that $x_i^*, p^*, p' > 0$ [12], [24], we upper-bound $\rho(C)$ with $||C||_\infty$, which for $c_{ik} > 0$ leads to $\rho(C) \leq \max_i(\sum_{k=1}^{N} c_{ik})$. This immediately yields (25). ■

It is easy to see that by letting $\alpha = \alpha_i$ and $\beta = \beta_i$ for all i, Theorem 7 directly translates to the sufficient condition of [24, Theorem 3].

F. Discussion

In this subsection, we verify the obtained results using Matlab simulations. Our first step is to check sufficiency and also lack of necessity in the condition of Theorem



(a) stable (b) unstable

Fig. 4. Delayed stability as a function of $\rho(A)$ and $\inf_W ||WAW^{-1}||_2$.

1. We generate 3000 two-by-two matrices and plot points (x, y) on a 2D plane, where $x = \rho(A)$ and $y = ||A||_2$. To detect instabilities, each matrix is tested with 100 random combinations of delay D_1 and D_2, each uniformly distributed in $[1, 30]$. We exclude all matrices with $\rho(A) > 1$ since these are a-priori known to be unstable. Out of 3000 random matrices, 1020 had $\rho(A) < 1$, out of which 468 were stable and 552 unstable under directional delay. Fig. 3(a) shows the stable points and (b) plots the unstable ones. From the first figure, notice that condition $||A||_2 \geq \rho(A)$ is never violated and Theorem 1 is *not* necessary for stability. At the same time, all unstable points in figure 3(b) are located above $||A||_2 = 1$, confirming the sufficiency of this condition.

Out of 468 stable matrices, 251 had $||A||_2 \geq 1$ and 331 had $||A||_\infty \geq 1$. Furthermore, 240 matrices had both norms above 1 simultaneously and in 87% of the cases, $||A||_2$ was smaller than $||A||_\infty$. Out of 552 unstable matrices, all had $||A||_2 \geq 1$ and $||A||_\infty \geq 1$. Moreover, 86% of the cases had $||A||_2 < ||A||_\infty$. It thus appears that $||A||_2$ is a tighter norm in terms of obtaining the necessary and sufficient condition.

The next simulation generates 10000 random two-by-two matrices and examines whether condition $||A||_s = \inf_W ||WAW^{-1}||_2 < 1$ is in fact sufficient for stability of the delayed system. Fig. 4 plots 3535 stable/unstable points (1763 stable and 1772 unstable). The largest $||A||_s$ for a stable matrix was 0.9953 and the smallest for an unstable matrix was 1.0024. This demonstrates that for the *generated* matrices, condition $||A||_s < 1$ was both sufficient and necessary. We leave further investigation of necessity of this condition to future work.

IV. Conclusion

In this paper, we studied delay-independent stability of max-min congestion control systems under heterogeneous diagonal delays. Our results improved the current understanding of conditions that A must satisfy from symmetry to diagonal similarity to matrices whose $||.||_2$ norm is less than one to ensure stability under arbitrary diagonal delays D_i (or, equivalently, directional delays $D_i^{\rightarrow}$ and $D_i^{\leftarrow}$). Although derived in the context of Internet congestion control, the obtained results are of broader interest and apply to any system that can be represented by model (1). As mentioned in the last section, our future work involves finding stability conditions that are both sufficient and necessary and identifying a wider class of systems that are stable under diagonal delays D_i as long as $\rho(A) < 1$.

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